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Lifshitz-like Janus Solutions

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Abstract

We construct Lifshitz-like Janus solutions in the Einstein-scalar theory with cosmological constant in arbitrary dimensions. They are holographically dual to $z = 2$ Lifshitz-like field theories with a defect. The four-dimensional solutions can be embedded into type IIB supergravity as dilatonic deformations of $AdS_5 \times Y^5$ with three-form field strengths, where Y^5 is a five-dimensional Einstein manifold.

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1 Introduction

Since the application of the AdS/CFT correspondence [1, 2] to the condensed matter physics has started, it is of great importance to find gravitational backgrounds dual to certain non-relativistic systems.

The first example is presented in [3, 4] with the form:

$$ds^2 = -r^{2z}(dx^-)^2 + 2r^2dx^+dx^- + r^2d\vec{x}^2 + \frac{dr^2}{r^2}, \quad (1.1)$$

where x^+ is compactified and x^- is regarded as the time direction. This geometry exhibits anisotropic scaling symmetry $x^- \equiv t \rightarrow \lambda^z t$, $x^+ \rightarrow \lambda^{2-z}x^+$, $x^i \rightarrow \lambda x^i$ and $r \rightarrow r/\lambda$. The parameter z is called a dynamical exponent and the symmetry of the geometry is enhanced to the Schrödinger symmetry including special conformal transformation when $z = 2$. Soon after that this type of the geometry with $z = 2$ was embedded into type IIB supergravity [5–7] deforming $AdS_5 \times S^5$ spacetime,¹ and the dual field theory is identified with a discrete light cone quantization (DLCQ) of the $\mathcal{N} = 4$ super Yang-Mills theory.

¹See also [8] for the embedding into the massive IIA theory.

The second example is a Lifshitz geometry of the form [9]²:

$$ds^2 = -r^{2z}dt^2 + r^2d\vec{x}^2 + \frac{dr^2}{r^2}. \quad (1.2)$$

This has not the Schrödinger symmetry but a part of it, namely, the non-relativistic scale invariance: $t \rightarrow \lambda^z t$, $x^i \rightarrow \lambda x^i$ and $r \rightarrow r/\lambda$. This solution can be constructed in the Einstein theory coupled to massive gauge fields with a negative cosmological constant [9,11]

Despite its simple form it is difficult to embed the four-dimensional Lifshitz solution into massive type IIA/M-theories with warped compactification [12, 13].³ Recently the four-dimensional Lifshitz solution with $z = 2$ was obtained by deforming the AdS_5 space in null coordinates in type IIB supergravity [16, 17]:

$$ds^2 = -2r^2dx^+dx^- + r^2d\vec{x}^2 + h(x^+)(dx^+)^2 + \frac{dr^2}{r^2}, \quad (1.3)$$

where we assume the five-dimensional internal manifold Y^5 is Einstein, *i.e.*, $R_{ij} = 4g_{ij}$. When the dilaton is constant, this is just the AdS space in the null coordinates, which is one realization of the Schrödinger geometry [5]. On the other hand, when the dilaton $\phi(x^+)$ depends on the null circle x^+ the function h is given by $h(x^+) = (\partial_+\phi(x^+))^2/4$ after solving the Einstein equation. This geometry does not have the Galilean boost symmetry $x_i \rightarrow x_i - v_i x^-$, $x^+ \rightarrow x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$ due to the g_{++} component of the metric. Thus the original Schrödinger symmetry breaks to the Lifshitz symmetry. After taking the Kaluza-Klein reduction along x^+ and identifying x^- with the time coordinate, it becomes the four-dimensional Lifshitz geometry (Lif_4) with $z = 2$, although it depends on the internal coordinate x^+ and should be viewed as a higher-dimensional geometry. More realistic solutions with h constant have been constructed by introducing the RR two-form and B-field in [17], which can be regarded as $Lif_4 \times S^1 \times Y^5$. Similar solutions with $z = 3$ and more general solutions with $z \geq 1$ are obtained in [18] and [19], respectively.

In this paper we would like to present a holographic description of the Lifshitz-like field theory with a defect across which the gauge coupling jumps. Our motivation comes from a junction of two systems which are possible to realize in condensed matter physics. There appear interesting phenomena in junction systems such as Andreev reflection that is a particle scattering which occurs at an interface between a superconductor and normal state material, and Josephson junction that is the phenomena of electric current between two superconductors separated by a thin insulator. We expect that this work will provide new insight into such systems in terms of holographic duality in future.

The relativistic field theory with a defect can be described by the Janus solution that is a deformation of AdS_5 space with a spatially varying dilaton [20]. The non-relativistic

²See also [10].

³ The five-dimensional Lifshitz solution was obtained in type IIB supergravity [14] with the dilaton depending on r . The schematic argument is given in [15].

Janus with the Schrödinger symmetry is also constructed by the null Melvin twist of the relativistic Janus in [21]. Further developments of the Janus solution are found in [23–43]. We would like to construct a generalized Janus solution with a Lifshitz-like asymptotic behavior near the boundary.

We will start with the construction of the Lifshitz-like Janus geometry in the type IIB supergravity by deforming AdS space with the varying dilaton, RR two-form and B-field. Actually we find the solutions of the following form:

$$ds^2 = f(\mu) \left(d\mu^2 + \frac{dz^2}{z^2} + z^2(-2dx^+dx^- + (dx_i)^2) \right) \\ + h(\mu, x^+)(dx^+)^2 + 2g(\mu, x^+)d\mu dx^+, \quad (1.4)$$

where the function $g(\mu, x^+)$ depends on the scalar field $\phi(\mu, x^+)$ and the function $f(\mu)$ is the same as that of the pure Janus solution. The function $h(\mu, x^+)$ obeys a partial differential equation that cannot be solved analytically. Therefore we focus on the special cases when it reduces to a ordinary differential equation. One may worry that our solutions have an extra $g_{\mu+}$ component different from the Lifshitz geometry, but it in fact falls off at the boundary and the metric approaches to (1.3). To see this, we need a coordinate transformation $(r, y) = (z \sec \mu, \sin \mu/z)$ (y is one of the spatial direction) and use the fact that $f(\mu) = 1/\cos^2 \mu$ when $g(\mu, x^+) = 0$. The scaling symmetry is obviously realized as $x^- \rightarrow \lambda^2 x^-$, $x_i \rightarrow \lambda x_i$, $z \rightarrow z/\lambda$. On the other hand our solutions reduce to the Janus geometry when the dilaton is independent of x^+ and the two-forms are set to zero. Thus our solutions have two expected aspects to describe the Lifshitz-like field theory with a defect at $y = 0$. Although the Lifshitz solution constructed in [17] keeps two supersymmetries when Y^5 is a Sasaki-Einstein manifold, our solution has no supersymmetry due to the dilatonic deformation.

The organization of this paper is as follows: we start with reviewing the constructions of the Janus geometries of type IIB supergravity in section 2. In section 3 we construct the Lifshitz-like Janus solutions in type IIB supergravity. We obtain a partial differential equation determining the function $h(\mu, x^+)$. We consider two special cases where we can solve it by an ordinary differential equation. In section 4 we also give an effective theory description of our solutions. We show that our construction can be generalized to arbitrary dimensions with the use of the Einstein-scalar theory. We discuss the dual field theories to our solutions in section 5. We conclude this paper with a discussion in section 6.

2 Review of Janus solutions

We would like to review a construction of the Janus solution in type IIB supergravity which is a dilatonic deformation of the AdS_5 spacetime [20]. This solution does not have supersymmetry, but is still stable non-perturbatively [22].

We start with summarizing the coordinates which are suitable for the Janus geometry, and move onto its construction.

2.1 Coordinates

The $(D + 1)$ -dimensional AdS space can be realized as a hyperboloid

$$X_0^2 + X_{D+1}^2 - \sum_{i=1}^D X_i^2 = 1, \quad (2.1)$$

in the flat $\mathbb{R}^{2,D}$ space with the metric

$$ds^2 = -dX_0^2 - dX_{D+1}^2 + \sum_{i=1}^D dX_i^2. \quad (2.2)$$

Obviously AdS_{D+1} geometry has an $SO(2, D)$ isometry. Parametrizing $X_D = \tan \mu$, the rest coordinates $X_{0,\dots,D-1,D+1}$ represent the AdS_D space with a radius $\sec \mu$. In other words, AdS_{D+1} space can be sliced by AdS_D space. From this, the metric can be written as

$$ds_{AdS_{D+1}}^2 = \sec^2 \mu (d\mu^2 + ds_{AdS_D}^2), \quad (2.3)$$

where $\mu \in [-\pi/2, \pi/2]$. This coordinate is useful to represent the Janus geometry as we will see below.

In the Poincaré coordinate, the metric is given by

$$ds_{AdS_{D+1}}^2 = -r^2 dt^2 + r^2 \sum_{i=1}^{D-1} dx_i^2 + \frac{dr^2}{r^2}, \quad (2.4)$$

and two coordinates (2.3) and (2.4) are connected with the relations ($r = z \sec \mu$, $x_{D-1} = \sin \mu / z$). Then the boundary $r = \infty$ in the Poincaré coordinate is divided into two parts $\mu = \pm \pi/2$ in the coordinates (2.3). Accordingly, the dual CFT to the AdS space can be defined on each boundary $x_{D-1} < 0$ and $x_{D-1} > 0$, respectively, in this coordinate.

2.2 Janus solutions

In the type IIB supergravity, the ansatz for the Janus solution takes the form,

$$ds^2 = f(\mu) (d\mu^2 + ds_{AdS_4}^2) + ds_{Y^5}^2, \quad (2.5)$$

$$\phi = \phi(\mu), \quad (2.6)$$

$$F_5 = 4 \left(f(\mu)^{\frac{5}{2}} d\mu \wedge \omega_{AdS_4} + \omega_{Y^5} \right), \quad (2.7)$$

where Y^5 is a five-dimensional Einstein manifold with $R_{ij} = 4g_{ij}$.⁴ ω_{AdS_4} and ω_{Y^5} are the volume forms of AdS_4 in unit radius and Y^5 , respectively. The metric is represented in the Einstein frame. When $f(\mu) = \sec^2 \mu$ and $\phi = \text{const.}$, it corresponds to the standard $AdS_5 \times Y^5$ solution. The dilatonic deformation breaks $SO(2, 4)$ isometry of AdS_5 to $SO(2, 3)$ of AdS_4 . The type IIB supergravity equations of motion are given by

$$R_{MN} - \frac{1}{2} \partial_M \phi \partial_N \phi - \frac{1}{96} F_{MN}^2 = 0, \quad (2.8)$$

$$\partial_M (\sqrt{-g} g^{MN} \partial_N \phi) = 0, \quad (2.9)$$

$$*F_5 = F_5. \quad (2.10)$$

The equation of motion for the dilaton is easily solved,

$$\partial_\mu \phi(\mu) = \frac{c}{f(\mu)^{\frac{3}{2}}}, \quad (2.11)$$

with a constant of integration c . The Einstein equation yields,

$$2f'f' - 2ff'' = -4f^3 + \frac{c^2}{2} \frac{1}{f}, \quad (2.12)$$

$$12f^2 + f'f' + 2ff'' = 16f^3. \quad (2.13)$$

These two equations can be simplified to one equation,

$$f'^2 = 4f^3 - 4f^2 + \frac{c^2}{6f}. \quad (2.14)$$

This equation is equivalent to the motion of a particle in the potential $V(f) = -(4f^3 - 4f^2 + c^2/6f)$ with zero energy. To keep the factor f in the metric to be positive, the top of the potential $V(f)$ must be positive. It requires $0 \leq c \leq 9/4\sqrt{2}$. Then (2.14) can be integrated,

$$\mu = \pm \frac{1}{2} \int_{f_{\min}}^f d\tilde{f} \left(\tilde{f}^3 - \tilde{f}^2 + \frac{c^2}{24} \frac{1}{\tilde{f}} \right)^{-\frac{1}{2}}, \quad (2.15)$$

where f_{\min} is the largest root of the equation $x^3 - x^2 + c^2/(24x) = 0$.

⁴In [20], S^5 was chosen as the internal space but it can be generalized with such a Einstein space Y^5 in the same manner.

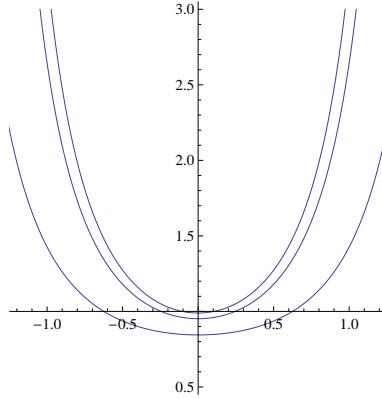


Figure 1: $f(\mu)$ as a function of μ for $c = 0.5, 1, 1.5$.

Note that taking the limit $f \rightarrow \infty$, $|\mu|$ approaches to a finite constant μ_c . Thus the range of μ is finite: $\mu \in [-\mu_c, \mu_c]$. The boundary values $\mu = \mu_c$ and $\mu = -\mu_c$ correspond to two halves of the boundary of the Janus geometry. Interestingly the dilaton approaches to two different constant values $\phi(\pm\mu_c) = \pm\phi(\mu_c)$ at the two halves as depicted in Figure 2. It makes the gauge coupling of the dual CFT jump across a defect inserted at $x_3 = 0$.

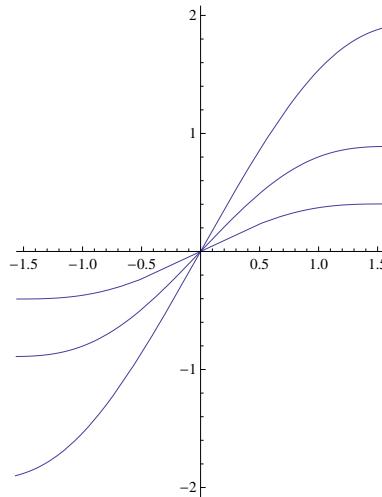


Figure 2: $\phi(\mu)$ as a function of μ for $c = 0.5, 1, 1.5$.

3 Lifshitz-like Janus solutions in type IIB supergravity

Now we describe Lifshitz-like generalization of the Janus solution in type IIB supergravity. Firstly, we start with general cases under the assumption of the appropriate metric form. Then we obtain a partial differential equation from the Einstein equation, but it cannot be solved analytically. Therefore, we focus on two cases where it reduces to an ordinary differential equation. Our solutions will become the Lifshitz geometries constructed in [16]

for the case 1 and a part of those given in [17] for the case 2, respectively.

3.1 Generalities

We set the axion zero for simplicity. In the Einstein frame, the action including a dilaton, B-field, RR three-form and RR five-form is

$$I = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi}|H_3|^2 - \frac{1}{2}e^\phi|F_3|^2 - \frac{1}{4}|\tilde{F}_5|^2 \right) - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 , \quad (3.1)$$

where $\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. The equations of motion are

$$\square\phi = -\frac{1}{12}(H_3)^2 e^{-\phi} + \frac{1}{12}(F_3)^2 e^\phi , \quad (3.2)$$

$$R_{MN} - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{12}e^{-\phi} \left(3(H_3^2)_{MN} - \frac{1}{4}(H_3)^2 g_{MN} \right) - \frac{1}{12}e^\phi \left(3(F_3^2)_{MN} - \frac{1}{4}(F_3)^2 g_{MN} \right) - \frac{1}{96}(\tilde{F}_5^2)_{MN} = 0 . \quad (3.3)$$

The fluxes satisfy the Bianchi identities:

$$dH_3 = dF_3 = d\tilde{F}_5 - H_3 \wedge F_3 = 0 , \quad (3.4)$$

and the equations of motion:

$$d(e^{-\phi} * H_3) = 0 , \quad d(e^\phi * F_3) = 0 , \quad d * \tilde{F}_5 = H_3 \wedge F_3 , \quad (3.5)$$

and the self-dual condition for \tilde{F}_5

$$\tilde{F}_5 = * \tilde{F}_5 . \quad (3.6)$$

We assume solutions with the following form:

$$ds^2 = f(\mu) \left(d\mu^2 + \frac{dz^2}{z^2} + z^2(-2dx^+dx^- + dx_1^2) \right) + h(\mu, x^+)(dx^+)^2 + 2g(\mu, x^+)dx^+d\mu + ds_{Y^5}^2 , \quad (3.7)$$

$$\phi = \phi(\mu, x^+) , \quad (3.8)$$

$$H_3 = dx^+ \wedge W_1 , \quad (3.9)$$

$$F_3 = dx^+ \wedge W_2 , \quad (3.10)$$

$$\tilde{F}_5 = 4(\omega_5 + \omega_{Y^5}) , \quad (3.11)$$

where Y^5 is a five-dimensional Einstein manifold with $R_{ij} = 4g_{ij}$ and W_a ($a = 1, 2$) are two-forms defined on Y^5 .

Given this ansatz, $(H_3)^2$ and $(F_3)^2$ vanish and the two-forms W_a satisfy:

$$dx^+ \wedge dW_a = d *_{Y^5} W_a = 0 . \quad (3.12)$$

These conditions are satisfied when W_a are harmonic two-forms on Y^5 .

The equation of motion for the dilaton is easily solved,

$$\partial_\mu \phi(\mu, x^+) = \frac{c(x^+)}{f(\mu)^{\frac{3}{2}}} , \quad (3.13)$$

with introducing a new function $c(x^+)$. The $\mu\mu$, $+-$, zz and yy components of the Einstein equation give the equations that resemble the Janus case.

$$2f'f' - 2ff'' = -4f^3 + \frac{c^2(x^+)}{2} \frac{1}{f} , \quad (3.14)$$

$$12f^2 + f'f' + 2ff'' = 16f^3 . \quad (3.15)$$

It follows from the first equation that the function $c(x^+)$ must be constant: $c(x^+) = c$. Thus $f(\mu)$ is exactly given by (2.15) and ϕ can be represented as

$$\phi(\mu, x^+) = c \int \frac{d\mu}{f(\mu)^{\frac{3}{2}}} + \phi^{(+)}(x^+) . \quad (3.16)$$

The range of μ and the possible value of c are the same as the Janus case.

The $+\mu$ component of Einstein equation determines the function g as

$$g(\mu, x^+) = \frac{c}{6} \frac{\partial_+ \phi(x^+)}{f(\mu)^{\frac{1}{2}}} . \quad (3.17)$$

Then $g(\mu, x^+)$ becomes zero at the boundary $\mu = \pm\mu_c$ as shown in Figure 3.

The $+z$ component is automatically satisfied from this solution.

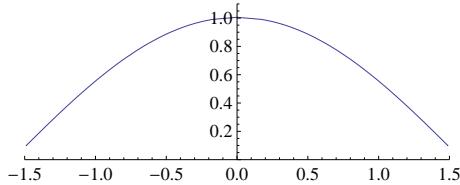


Figure 3: $f(\mu)^{-\frac{1}{2}}$ as a function of μ for $c = 0.5$, to which $g(\mu)$ is proportional.

The $++$ component will be a differential equation for h :

$$\begin{aligned} & \frac{\partial_\mu^2 h(\mu, x^+)}{2f} - \frac{f'}{4f^2} \partial_\mu h(\mu, x^+) - \left(2 - \frac{c^2}{12f^4} \right) h(\mu, x^+) \\ & + \left(\frac{1}{2} - \frac{c^2}{18f^3} \right) (\partial_+ \phi^{(+)})^2 + \frac{(W_1)_{ij}(W_1)^{ij}}{4} e^{-\phi(\mu, x^+)} + \frac{(W_2)_{ij}(W_2)^{ij}}{4} e^{\phi(\mu, x^+)} = 0 . \end{aligned} \quad (3.18)$$

Given the parameter c and the harmonic two-forms W_α , we can solve this partial differential equation and obtain the function $h(\mu, x^+)$ numerically. The term that involves dx^+ in the metric (3.7) can be written as

$$h(\mu, x^+) \left(dx^+ - z^2 \frac{f(\mu)}{h(\mu, x^+)} dx^- + \frac{g(\mu, x^+)}{h(\mu, x^+)} d\mu \right)^2 - \frac{1}{h(\mu, x^+)} (z^2 f(\mu) dx^- - g(\mu, x^+) d\mu)^2 . \quad (3.19)$$

Compactification of the first bracket direction yields the Lifshitz-like geometry with the dynamical exponent $z = 2$ near the boundary. More precisely, our solution has a scaling symmetry $t \equiv x^- \rightarrow \lambda^2 x^-$, $x^1 \rightarrow \lambda x^1$, $z \rightarrow z/\lambda$ with x^+ fixed, where x^- is identified with a new time coordinate. We, however, will focus on the special cases such that the analytic solutions can be constructed below.

3.2 Case 1: $W_a = 0$

We can simplify the partial differential equation (3.18) when $W_a = 0$, and reduce it to the ordinary differential equation further requiring $h(\mu, x^+) = \tilde{h}(\mu)(\partial_+ \phi^{(+)})^2$:

$$\frac{\tilde{h}''}{2f} - \frac{f'}{4f^2} \tilde{h}' - \left(2 - \frac{c^2}{12f^4} \right) \tilde{h} + \left(\frac{1}{2} - \frac{c^2}{18f^3} \right) = 0 . \quad (3.20)$$

Expanding \tilde{h} in $1/f$ as

$$\tilde{h}(\mu) = \sum_{k=k_0}^{\infty} \frac{a_k}{f(\mu)^k} , \quad (3.21)$$

and using (3.14), (3.15), and (3.20), we can obtain an equation without the differentials of f ,

$$\begin{aligned} & \left(\frac{1}{2} - \frac{c^2}{18f^3} \right) + 2a_{k_0}(k_0^2 - 1)f^{-k_0} \\ & + (-a_{k_0}k_0(2k_0 + 1) + 2a_{k_0+1}((k_0 + 1)^2 - 1))f^{-(k_0+1)} \\ & + (-a_{k_0+1}(k_0 + 1)(2(k_0 + 1) + 1) + 2a_{k_0+2}((k_0 + 2)^2 - 1))f^{-(k_0+2)} \\ & + (-a_{k_0+2}(k_0 + 2)(2(k_0 + 2) + 1) + 2a_{k_0+3}((k_0 + 3)^2 - 1))f^{-(k_0+3)} \\ & + \sum_{k=k_0}^{\infty} \left(\frac{c^2 a_k}{12} (k+1)^2 - a_{k+3}(k+3)(2(k+3)+1) + 2a_{k+4}((k+4)^2 - 1) \right) f^{-(k+4)} = 0 . \end{aligned} \quad (3.22)$$

It determines the power of the leading term $k_0 = 0, -1$ and the coefficients a_i are determined recursively. Thus there are two types of solutions near the boundary of μ where $f(\mu)$ diverges. One is proportional to f , *i.e.*, $k_0 = -1$ (Figure 4):

$$\tilde{h}(\mu) \simeq \alpha f(\mu) \quad (\mu \rightarrow \pm \mu_c) , \quad (3.23)$$

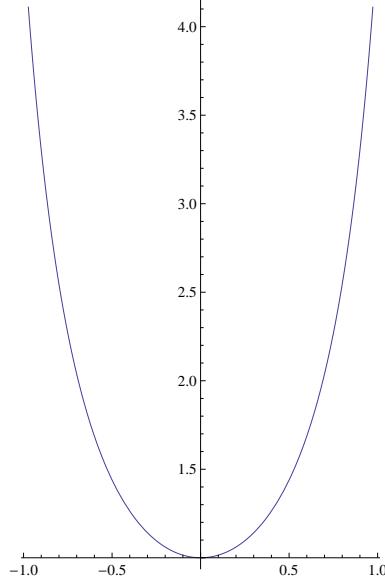


Figure 4: $\tilde{h}(\mu)$ is proportional to $f(\mu)$ at the boundary as a function of μ for $c = 0.5$.

and the other approaches constant, *i.e.*, $k_0 = 0$, $a_{k_0} = 1/4$ (Figure 5):

$$\tilde{h}(\mu) \simeq \frac{1}{4} \quad (\mu \rightarrow \pm\mu_c) . \quad (3.24)$$

In the latter case, the function $h(\mu, x^+)$ approaches to $(\partial_+\phi^{(+)}(x^+))^2/4$, and the function

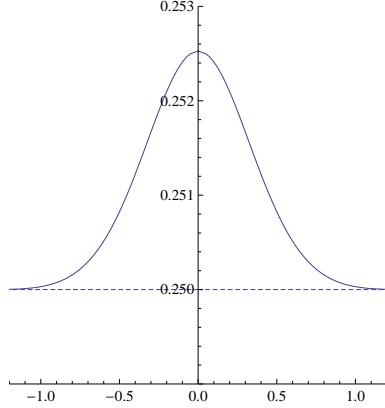


Figure 5: $\tilde{h}(\mu)$ approaches to $1/4$ at the boundary as a function of μ for $c = 0.5$.

$g(\mu, x^+)$ given by (3.17) is going to zero near the boundary. Then our solution actually behaves like the Lifshitz solution given by [4], while the dilaton takes different values on each half of the boundary similarly to the Janus solution. Therefore we realize the Lifshitz-like Janus solution in our setup.

3.3 Case 2: $\partial_+\phi^{(+)} = 0$

The second example is a solution with $\partial_+\phi^{(+)} = 0$, when $g(\mu, x^+) = 0$ and the partial differential equation (3.18) reduces to an ordinary differential equation under the assumption such that h and W_a do not depend on x^+ .

Thus we can solve the equation once the parameter c is given. When $c = 0$, the dilaton becomes constant and (3.18) has a simple solution $h = (W_1)_{ij}(W_1)^{ij}e^{-\phi}/8 + (W_2)_{ij}(W_2)^{ij}e^\phi/8$, which is included in the solutions given in [17]. When $c \neq 0$, we have a solution with following behaviors around the boundary:

$$h(\mu \rightarrow \pm\mu_c) = \frac{(W_1)_{ij}(W_1)^{ij}}{8}e^{-\phi(\pm\mu_c)} + \frac{(W_2)_{ij}(W_2)^{ij}}{8}e^{\phi(\pm\mu_c)}. \quad (3.25)$$

Therefore our solution is the Lifshitz-like Janus geometry with a constant h near the boundary. If we take $W_1 = W_2$, the value of h at each half of the boundary is equal because the dilaton $\phi(\mu)$ is the odd function of μ .

As opposed to the previous case, Y^5 cannot be arbitrary but should be a manifold which has a non-trivial two-cycle to define the harmonic two-forms on it. The concrete example is the Sasaki-Einstein space $T^{1,1}$ [44, 45] with the metric of the form:

$$ds_{T^{1,1}}^2 = \frac{1}{6} \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2, \quad (3.26)$$

where $\psi \in [0, 4\pi]$. It is obvious that $T^{1,1}$ is a S^1 fibration on $S^2 \times S^2$. We can choose the harmonic two-forms on $T^{1,1}$ as

$$W_1 = W_2 = \frac{w}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2), \quad (3.27)$$

where w is a constant we can tune freely. Then the value of h at the boundary is

$$h(\mu = \pm\mu_c) = 9w^2 \cosh \phi(\mu_c), \quad (3.28)$$

and the radii of the circle x^+ are equal on each boundary.

4 Effective theory for Lifshitz-like Janus solutions

We have constructed the Lifshitz-like Janus solutions in type IIB supergravity so far. It is worthwhile to consider generalization of our solutions to arbitrary dimensions in effective theories, although we don't know how they can be embedded into the string/M-theories.

We consider the Einstein-Hilbert action with one neutral and one complex scalars assuming a similar form of the metric. The role of the B- and RR-fields is played by the complex scalar field which gives the energy-momentum tensor with only ++ component.⁵

⁵We are grateful to C. Herzog for useful discussions of this construction.

4.1 Einstein-scalar theory

The key to obtain the Lifshitz-like solution is introducing the energy-momentum tensor with only ++ component. Let us consider Lifshitz-like Janus solutions in the $(D+1)$ -dimensional Einstein-Hilbert theory with one neutral and one complex scalars

$$I = \int d^{D+1}x \sqrt{-g} \left(R + \Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}|\partial\chi|^2 \right) , \quad (4.1)$$

When $\Lambda = D(D-1)/\ell^2$ there exists an AdS_{D+1} solution with a curvature radius ℓ with constant scalar fields.

The equations of motion are the followings:

$$R_{MN} = -Dg_{MN} + \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{4}(\partial_M\chi\partial_N\bar{\chi} + \partial_M\bar{\chi}\partial_N\chi) , \quad (4.2)$$

$$\square\phi = \square\chi = 0 . \quad (4.3)$$

The Lifshitz-like Janus solution we consider is a deformation of the AdS_{D+1} space with varying dilaton and we take the following ansatz setting $\ell = 1$:

$$ds^2 = f(\mu) \left(d\mu^2 + \frac{dz^2}{z^2} + z^2(-2dx^+dx^- + (dx_i)^2) \right) + h(\mu, x^+)(dx^+)^2 + 2g(\mu, x^+)d\mu dx^+ , \quad (4.4)$$

$$\phi = \phi(\mu, x^+) , \quad (4.5)$$

$$\chi = \chi(x^+) , \quad (4.6)$$

where i runs from 1 to $D-3$. The equation of motion for the complex scalar is automatically satisfied.

This situation is very similar to the previous case once W is replaced with χ . The complex scalar field contributes to only the ++ component of the energy-momentum tensor. Then we obtain Lifshitz-like Janus solutions similarly to the type IIB case.

The components of the Ricci tensor we will use below are as follows

$$R_{\mu\mu} = \frac{D(f'^2 - ff'')}{2f^2} , \quad (4.7)$$

$$R_{zz} = -\frac{2ff'' + (D-3)f'^2 + 4(D-1)f^2}{4z^2f^2} , \quad (4.8)$$

$$R_{\mu+} = -\frac{g((D-3)f'^2 + 2ff'')}{4f^3} , \quad (4.9)$$

$$R_{z+} = -\frac{2f\partial_\mu g + (D-3)g\partial_\mu f}{2zf^2} , \quad (4.10)$$

$$R_{++} = -\frac{1}{4f^3} (2f^2\partial_\mu^2 h - (5-D)ff'\partial_\mu h + 2(f'^2 + 4f^2)h - f(8g^2 + 2(D-3)f'\partial_\mu g + 4f\partial_\mu^2 g)) , \quad (4.11)$$

$$R_{+-} = -z^4 R_{zz} , \quad R_{ii} = z^4 R_{zz} . \quad (4.12)$$

The ansatz (4.6) automatically solves the equation (4.3) for the complex scalar. The equation (4.3) for the real scalar is solved

$$\phi(\mu, x^+) = c(x^+) \int \frac{d\mu}{f(\mu)^{\frac{D-1}{2}}} + \phi^{(+)}(x^+) . \quad (4.13)$$

The $\mu\mu$ part and the $+-$, zz and ii components of the Einstein equation give the equations

$$D(f'^2 - ff'') = -2D f^3 + \frac{c(x^+)^2}{f^{D-3}} , \quad (4.14)$$

$$4(D-1)f^2 + (D-3)f'^2 + 2ff'' = 4D f^3 . \quad (4.15)$$

It follows from the first equation that $c(x^+)$ must be constant and thus the function $f(\mu)$ is obtained by the following relation

$$c(x^+) = c = \text{const} , \quad (4.16)$$

$$\mu = \pm \frac{1}{2} \int_{f_{\min}}^f d\tilde{f} \left(\tilde{f}^3 - \tilde{f}^2 + \frac{c^2}{2D(D-1)\tilde{f}^{D-3}} \right)^{-\frac{1}{2}} , \quad (4.17)$$

where f_{\min} is given by the largest root of the equation $x^3 - x^2 + c^2/(2D(D-1)x^{D-3}) = 0$. In the same manner as the last section, the parametric range of c is

$$0 \leq c \leq \sqrt{\frac{2(D-1)^D}{D^{D-1}}} . \quad (4.18)$$

The $\mu+$ part of the Einstein equation is

$$-\frac{g}{4f^3} ((D-3)f'^2 + 2ff'') = -Dg + \frac{1}{2}\partial_\mu\phi\partial_+\phi , \quad (4.19)$$

and it can be simplified by using (4.13) and (4.15) to

$$g(\mu, x^+) = \frac{c\partial_+\phi^{(+)}}{2(D-1)f^{\frac{D-3}{2}}} . \quad (4.20)$$

The $z+$ component is automatically satisfied by the above solution.

Finally we have to solve the $++$ component but it is a partial differential equation.

$$\begin{aligned} \frac{1}{2f}\partial_\mu^2 h - \frac{(5-D)}{4f^2}f'\partial_\mu h + \left((2-D) + \frac{c^2}{D(D-1)f^D} \right) h \\ + \left(\frac{1}{2} - \frac{c^2}{2(D-1)^2f^{D-1}} \right) (\partial_+\phi^{(+)})^2 + \frac{1}{2}|\partial_+\chi|^2 = 0 . \end{aligned} \quad (4.21)$$

Solving this differential equation, we obtain Lifshitz-like Janus solutions in arbitrary dimensions. Corresponding to the special cases in type IIB supergravity, it can be reduced to an ordinary differential equation when appropriate fields are set to zero. It is worth

mentioning that the function $g(\mu, x^+)$ goes to zero near the boundary when $D \geq 4$, and then our solution actually behaves as a $(D + 1)$ -dimensional Lifshitz solution.

In the case of $\partial_+\chi = 0$ the above equation is simplified to an ordinary differential equation with $h(\mu, x^+) = \tilde{h}(\mu)(\partial_+\phi^{(+)})^2$. When $c = 0$ this solution becomes a Lifshitz-like geometry with $f(\mu) = \sec^2 \mu$ and

$$\tilde{h}(\mu) = \frac{1}{2(D - 2)} . \quad (4.22)$$

When $c \neq 0$ we still have a solution which approaches the above value near the boundary $\mu = \pm\mu_c$.

On the other hand, we have another class of solutions when $\partial_+\phi^{(+)} = 0$ corresponding to the case 2 considered in section 3.3. We need to take $|\partial_+\chi|^2$ constant to reduce (4.21) to an ordinary differential equation of h with respect to μ . Then the boundary value of h is

$$h(\mu = \pm\mu_c) = \frac{|\partial_+\chi|^2}{2(D - 2)} . \quad (4.23)$$

Notice that g is zero in this case and χ must be complex to take $|\partial_+\chi|^2$ constant since x^+ is compactified to S^1 .

5 Dual field theories

The $AdS_5 \times S^5$ background with a constant dilaton in type IIB supergravity is dual to the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with an appropriate gauge coupling. Once we deform the background with varying the dilaton, the gauge coupling in the dual field theory depends on the spacetime coordinates. More precisely the relation between the gauge coupling and the dilaton is given by $g_{YM}^2(x^M) = e^{\phi(x^M)}$ where x^M denotes the spacetime coordinates at the boundary.

The Lifshitz-like solution with $\phi = \phi(x^+)$ is a light-like deformation of the AdS_5 with the nontrivial dilaton $\phi(x^+)$. This solution exhibits anisotropic scaling symmetry with the dynamical exponent $z = 2$ such that $x^- \rightarrow \lambda^2 x^-$, $x_i \rightarrow \lambda x_i$ and $z \rightarrow \lambda^{-1}z$. In addition to this dilatation symmetry \mathcal{D} , there are time translation \mathcal{H} , spatial translations \mathcal{P}^i and spatial rotations \mathcal{M}^{ij} . This symmetry algebra closes on itself. If the dilaton is constant, the background enjoys the Schrödinger symmetry including the Galilean boosts \mathcal{K}^i , special conformal transformation \mathcal{C} and mass or particle number operator \mathcal{M} . Now we don't have these extended symmetries due to the g_{++} component in the metric.

The Janus solution with $\phi = \phi(\mu)$ has the $SO(3, 2)$ symmetry which is the subgroup of the $SO(4, 2)$ symmetry of AdS_5 . The dilaton varies in the bulk and takes different values on each boundary $\mu = \pm\mu_c$. This means that the dual field theory is the $\mathcal{N} = 4$ SYM living

on the boundary consisting of two half spaces $\mu = \pm\mu_c$, where the coupling constants are given by $g_{YM}^2(\pm\mu_c) = e^{\phi(\pm\mu_c)}$. In the Poincaré coordinates ($r = z \sec \mu$, $x_3 = \sin \mu/z$), the two halves of the boundary are glued at $x_3 = 0$ and the gauge coupling jumps across a defect at $x_3 = 0$. Although the four-dimensional conformal symmetry is broken, there still exists three-dimensional conformal symmetry $SO(3, 2)$ transverse to the defect. In addition the dilatation symmetry enhances and acts as $t \rightarrow \lambda t$, and $x_i \rightarrow \lambda x_i$ as well as $x_3 \rightarrow \lambda x_3$. This symmetry algebra doesn't change the location of the defect and thus the dual field theory is called a defect CFT.

The $(D + 1)$ -dimensional Lifshitz-like Janus solution in the effective theory interpolates these two types of solutions and dual to the $(D - 1)$ -dimensional Lifshitz-like defect CFT with the dynamical exponent $z = 2$. In the type IIB context, the dual field theory for the case 1 is the DLCQ of $\mathcal{N} = 4$ SYM with a defect since x^+ coordinate is compactified, while we don't understand the specific structure of the dual field theory for the case 2. We will discuss it in section 6.

Before concluding this section, we would like to mention the difference between our solution and the non-relativistic Janus solution constructed in [21]. The situation is very similar, but the latter is dual to the defect CFT with the Schrödinger symmetry that is larger than that of our solutions.

6 Discussion

It seems to be possible to embed three-dimensional Lifshitz-like Janus solutions into M-theory since the action (4.1) we used is fairly simple. But it is actually hard to obtain a massless scalar field by compactifying IIA/M-theory on a six-dimensional internal manifold. Fluxes in type II theory prevents the dilaton from being massless in general although a self-dual flux doesn't. This is why we can find the Lifshitz-like Janus solution in type IIB case.

One may think such a solution can be obtained by deforming the $AdS_4 \times Y^7$ spacetime with the four-form flux

$$F_4 = 3\omega_4 + d\phi(x^+, \mu) \wedge W , \quad (6.1)$$

where W is a three-form defined on the internal manifold Y^7 . Unfortunately, this way of construction doesn't work as described below. The equations of motion and the Bianchi identity for the flux are satisfied if $\square\phi = 0$, $dW = 0$ and $d*_7 W = 0$. The former is the equation of motion for a massless scalar and the latter means that W is a harmonic three-form on Y^7 . On the other hand the internal component of the Einstein equation requires $(\partial\phi)^2 = 0$, but this cannot be hold on our background (4.5) unless ϕ is independent of μ , which finally reduces to the Lifshitz geometry given in [17] and the appendix A.1 of [16].

Our solutions are non-supersymmetric due to the dilatonic deformation. Then they could be unstable against perturbation. There exists supersymmetric Lifshitz [17] and Janus solutions [26, 27, 31, 32, 37, 40–42], respectively. Thus it must be interesting to investigate the construction of supersymmetric Lifshitz-like Janus solutions in string/M-theories. Our effective theory construction might be useful for this purpose.

In the case 2 with $Y^5 = T^{1,1}$ in section 3.3, our solution becomes a deformation of Klebanov-Witten theory [46] with the dilaton, the B- and RR 2-form fluxes on $T^{1,1}$. When the dilaton is constant, our solution reduces to that of [17], which consists of D3-branes on a conifold and (p, q) five-branes wrapped over a three-cycle in $T^{1,1}$. The remaining three dimensions of the (p, q) five-branes parallel to $\mathbb{R}^{1,3}$, and then it would represent a light-like domain wall in the dual field theory similarly to [47, 48]. The Lifshitz-like field theory is expected to be realized on this domain wall, although we have no evidence for it except the scaling symmetry of our bulk geometry. It will be interesting to investigate this further.

More interesting things can happen if there is a black hole inside our solutions. Dual field theories will be heated up and there would be phase transitions in the presence of matters. Unfortunately, black hole solutions have not been constructed even in the Janus geometry except the special case [30]. It is of great importance to construct such solutions in order to holographically describe more realistic field theories related with condensed matter physics.

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